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CHARACTERIZATION OF MAXIMAL IDEALS

by

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UNITED STATES
NAVAL POSTGRADUATE SCHOOL

THESIS

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OF MAXIMAL IDEALS

by

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September 1968

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Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE WITH MAJOR IN MATHEMATICS

from the

NAVAL POSTGRADUATE SCHOOL
September 1968

ABSTRACT

The characterization of maximal ideals has been accomplished for the class of principal ideal rings and for the class of continuous functions on a compact topological space. In the latter case the characterization is somewhat more extensive than it appears since a non-compact space can often be compactified. Closely related to maximal ideals are the prime ideals and the coincidence of these two types of ideals is noted where occurring. When the prime and maximal ideals are not the same in a ring, illustrative examples are given.

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ACKNOWLEDGEMENTS

I am indebted to my advisor, Professor E. S. Langford, for the encouragement and guidance given me. I also express my gratitude to the members of the department of mathematics of the Naval Postgraduate School who helped to make the study of mathematics a pleasant and fulfilling experience. Especially do I thank Professors E. J. Stewart and F. D. Faulkner whose cheerful attitude made their courses enjoyable as well as informative.

INTRODUCTION

The maximal ideals of Z, the ring of integers are easily seen to be all multiples of a prime number p. This simple characterization suggests the question of whether the maximal ideals of any ring can be so simply dealt with. One sees that an extension of the prime numbers of Z to more general prime elements in a principal ideal ring leads to a similar result for the class of principal ideal rings which includes the class of Euclidean rings. Additionally one sees that the prime and maximal ideals are identical in Z. This relationship too is extended to the class of principal ideal rings. One then notes that in C(X), the rings of continuous functions on a topological space X, no function satisfies the requirements for being a prime element. Indeed it is seen that the prime and maximal ideals are not the same and are not necessarily principal.

I. INTRODUCTORY CONCEPTS

In what follows R will be a commutative ring with identity, denoted by 1.

DEFINITION

A subset I of a ring R is said to be a (two-sided) ideal of R if:

(1) I is a subgroup of R under addition.

(2) For every $u \in I$ and $r \in R$, the product ru is in I, i.e. $rI \subseteq I$ for all $r \in R$.

An ideal I is proper if $I \neq R$. In any ring two examples of ideals are $\{0\}$ and R itself. These are called trivial ideals.

Example

(1) Consider the ring of integers Z. Let n be any integer. Let $I_n = nZ = Zn$, i.e. I_n is the subset of Z consisting of all possible multiples zn with $z \in Z$. Then I_n is an ideal.

(2) Let C(I) denote the ring of continuous functions on the unit interval I under the usual pointwise opeartions. Then $M_o = \{f \in C(I) : f(0) = 0\}$ is an ideal.

Lemma 1

The intersection of any family of ideals is an ideal. (Note: the intersection of the empty family is defined to be R.)

Proof. Let $\mathfrak{I}$ be any family of ideals of R. If $\mathfrak{I} = \emptyset$, by definition $\cap\mathfrak{I} = R$ is an ideal, so suppose that $\mathfrak{I} \neq \emptyset$.

Since $0 \in I$ for all $I \in \mathfrak{I}$, $F = \cap\mathfrak{I}$ is not empty. If $a,b \in F$, then $a,b \in I$ for every $I \in \mathfrak{I}$, consequently $a+b \in I$ for every $I \in \mathfrak{I}$. Hence $a+b \in F$. Similarly if $a \in F$ and $r \in R$, then $-a \in F$ and $ra \in F$ so that F is an ideal.

Lemma 2

If S is any subset of R, $\exists$ a unique smallest ideal I(S) which $\supseteq$ S.

Proof. Let $\mathfrak{I}$ denote the family of all ideals containing S and consider $\cap\mathfrak{I}$, i.e. the intersection of all members of $\mathfrak{I}$. From Lemma 1 this intersection is an ideal. It is obvious that any ideal containing S also contains $\cap\mathfrak{I}$. Therefore $I(S) = \cap\mathfrak{I}$.

Remark: I(S) is the ideal generated by S. If $S = \{a\}$ consists of a single point, we usually write $I(S) = (a)$. It is easy to see that $(a) = Ra = aR$. (if R is not commutative or does not possess an identity, then this may not be true). The ideal (a) is the ideal generated by a, and an ideal is principal if it can be generated by a single element.

Example

Let Z be the ring of integers and let n be any integer. The ideal I_n is principal for $I_n = (n)$. In fact, every ideal of Z is of this form. I_n is proper if and only if $n \neq \pm 1$.

DEFINITION

An ideal P is a prime ideal if $a \in P$ or $b \in P$ whenever $ab \in P$.

Example

The ideal (n) as in the previous example is prime if and only if $n = \pm p$, where p is a prime number.

DEFINITION

An ideal Q is a primary ideal if whenever $ab \in Q$, but $a \notin Q$, then $b^n \in Q$ for some integer $n \geq 1$.

Example

It is obvious that every prime ideal is primary. In the ring of integers, (n) is primary if and only if $n = \pm p^k$, where p is a prime number and k is an integer ≥ 1. That an ideal of this form is primary follows from the fact that $(p^k) \subseteq (p)$. For consider $ab \in (p^k)$, then ab is also in (p). If we assume that $a \notin (p)$, then b must be in (p) because (p) is prime. Since $b \in (p)$, b must be writable as $b = mp$ for some integer m. Now consider $b^k = m^k p^k$. Since b^k is writable as an integer multiple of p^k, and since ab was arbitrary, we have shown (p^k) to be primary.

To show that all the primary ideals of Z are of the form (p^k) recall the following:

(1) Any positive integer can be factored uniquely as the product of the power of prime numbers.

(2) If a is relatively prime to b, but $a \mid bc$, then $a \mid c$.

Now consider an integer n which is not the power of a prime. We shall assume the ideal (n) to be primary and obtain a contradiction. From (1) n may be written as $n = p^k q^m s$ where p and q are distinct primes, k and m are integers ≥ 1,

and where $(s,p) = (s,q) = 1$. Set $x = p^k$ and $y = q^m s$. Since $n \in (n)$, $xy \in (n)$, but obviously $x \notin (n)$. Having assumed (n) to be primary we must have that $y^j \in (n)$ for some $j \geq 1$. Therefore $y^j = rn$ for some integer r, i.e. $q^{mj}s^j = rp^k q^m s$. But p divides the right hand side so it must divide the left hand side. However $(s,p) = (q,p) = 1$ and p is prime so we have a contradiction. Therefore $y^j \notin (n)$ for any $j \geq 1$ and (n) is not primary.

Lemma 3

The union of any non-empty chain (i.e. nested family) of ideals is an ideal.

Proof. Let ζ denote a nested family of ideals and let $C = \cup\zeta$. Obviously $0 \in C$. Now let a, b be any two elements of C. Then $a \in I$ for some $I \in \zeta$ and similarly $b \in J$ for some ideal $J \in \zeta$. Since ζ is nested, either $J \subseteq I$ or $I \subseteq J$. Assume that $I \subseteq J$. Then $a, b \in J$, and hence $a+b \in J$ and $a+b \in C$. Similarly $a \in C$ implies that $-a \in C$ and $ra \in C$ for all $r \in R$. Therefore C is an ideal.

Lemma 4

If S and T are ideals, then the smallest ideal which contains S and T is $S+T = \{s+t : s \in S \text{ and } t \in T\}$. I.e. $S+T = I(S \cup T)$.

Proof. Obviously $S+T$ defined above is an additive subgroup. Let $s \in S$ and $t \in T$ with $r \in R$. Consider $r(s+t) = rs+rt$. Since $rs \in S$ and $rt \in T$, then $r(s+t) \in S+T$. Therefore $S+T$ is an ideal. It is obvious that $S+T$ contains

both S and T and that any ideal containing S and T will contain S+T; therefore S+T is the smallest ideal containing both S and T.

DEFINITION

A proper ideal M of R is a maximal ideal if there exist no proper ideals of R which contain M. I.e. if $M \subseteq I$ and I is proper then M = I.

Example

In the ring of integers, the maximal ideals are the same as the prime ideals, for consider an ideal (a) with a composite. Then a = nb for integers n, $b \neq 1$; one can see that $(b) \supset (a)$ and that (b) is proper. Therefore a composite means that (a) is not maximal.

Now let p be a prime and consider (p). The ideal (p) is maximal for assume $(a) \supset (p)$ where (a) is proper. Then $p \in (a)$ so that p = na for some integer $n \neq 1$. But this is impossible because p is prime. Therefore the ideal (p) is maximal in Z. (C.f. Lemma 10.)

We have been considering a ring in which the prime and maximal ideals are precisely the same. One might ask if this is true in every ring. We shall see that generally this is not so. There is however an important class of rings (of which the ring of integers is an example) in which an ideal is maximal if and only if it is prime. This is the class of principal ideal rings.

II. MAXIMAL AND PRIME IDEALS IN A PRINCIPAL IDEAL RING

DEFINITION

An element $a \neq 0$ of R is a zero-divisor if there exists a non-zero element $b \in R$ such that $ab = 0$.

Example

Consider the ring of integers modulo 6. The elements 2 and 3 are zero-divisors.

DEFINITION

A (commutative) ring is an integral domain if it has no zero-divisors. Thus a (commutative) ring is an integral domain if and only if it satisfies the cancellation law: $ax = bx$ and $x \neq 0$ imply $a = b$.

DEFINITION

An integral domain R (whith identity) is a principal ideal ring if every ideal I of R is of the form $I = (a)$ for some $a \in R$. As noted earlier the ring of integers is an example of a principal ideal ring.

DEFINITION

An integral domain R is an Euclidean ring if for every $a \neq 0$ in R there is defined a nonnegative integer $d(a)$ such that:

(1) For all $a, b \in R$, both nonzero, $d(a) \leq d(ab)$.

(2) For every $a, b \in R$, both nonzero, there exist $t, r \in R$ such that $a = tb+r$ where wither $r = 0$ or $d(r) < d(b)$.

Examples

(1) The ring of integers with $d(a) = |a|$.

(2) The ring of complex numbers of the form a+bi where a and b are integers, and where $d(a+bi) = a^2+b^2$. This is the ring of Gaussian integers.

(3) Let F be a field. Consider the ring of polynomials F[x]. The elements of this ring are of the form $f(x) = a_o + a_1x + a_2x^2 + \cdots + a_nx^n$, where the coefficients a_i ($i = 0, 1, \cdots, n$) are elements of the field F, and where either $f(x) = 0$ or $a_n \neq 0$. Let f(x) be any element of F[x]. Let $d(f) = n$, i.e. d(f) = the largest exponent of the "variable" x which has a non-zero coefficient. Then F[x] is an Euclidean ring.

Lemma 5

Let R be an Euclidean ring. Then R is a principal ideal ring.

Proof. Let I be any ideal of R. Choose $a \in I$ such that d(a) is minimal. Since d assumes only nonnegative integer values this can be done.

Now let b be any element of I. Since I is Euclidean, $b = ta+r$ with $r = 0$ or $d(r) < d(a)$. Since I is an ideal, $r \in I$. But a was chosen so that d(a) is minimal so that $r = 0$. Therefore $b = ta$. Since b was chosen arbitrarily every element of I is of this form. Hence I is the principal ideal (a) and R is a principal ideal ring.

One might inquire as to whether every principal ideal ting is also an Euclidean ring. T. Motzkin [1] has shown that this is false.

We now show that in any principal ideal ring, the prime and maximal ideals are the same and are generated by elements which are analogous to the prime numbers.

DEFINITION

An element $a \in R$ is a unit or invertible element if there exists an element $b \in R$ such that $ab = 1$. Recall that in a field every non-zero element is a unit.

Examples

(1) In Z there are no units other than ± 1.

(2) In $F[x]$ the units are the polynomials of degree $n = 0$.

(3) Consider C(I), the ring of continuous functions on the unit interval $[0,1]$. Let f be any function which is non-vanishing in I. Then $g = 1/f$ is in C(I) and $fg = 1$, so that f is a unit. We shall have a closer look at this ring a little later.

DEFINITION

An element $p \in R$ is a prime element if whenever $p = ab$ with $a, b \in R$, then a or b is a unit.

Examples

(1) We have already seen that in Z the prime elements are simply the prime numbers or their negatives.

(2) Let R be the field of real numbers. The element $f(x) = x^2 + 1 \in R[x]$ cannot be written as the product of two non-unit elements. Such a polynomial is called irreducible. The irreducible polynomials are the prime elements of R[x]. It is well known that a polynomial is irreducible if and only if there is no solution in R for r such that $f(r) = 0$.

(3) In C(I) there are no prime elements. Let f be any function in C(I). The functions $g = \sqrt{|f|}$ and $h = \operatorname{sgn} f\sqrt{|f|}$ are invertible only if f is invertible, and will be continuous since f is continuous. Therefore any non-unit, f, of C(I) may be expressed as f = gh with g and h non-units in C(I).

To show that the primary ideals of Z were generated by p^k with p a prime number and $k \geq 1$, we used the fact that any integer could be factored uniquely as the product of powers of primes. Having now introduced the more general prime element we extend some of the properties of prime numbers to prime elements.

DEFINITION

If $a \neq 0$ and b are elements of R, a divides b if there exists an element $c \in R$ such that b = ac. We write $a|b$ if a divides b and $a \nmid b$ if a does not divide b.

DEFINITION

Two elements a and b are associates if $a|b$ and $b|a$. This is obviously an equivalence relation on R. If $b \in R$ and if $u \in R$ is a unit then $u|b$, for $1b = uu^{-1}b = u(u^{-1}b)$.

Lemma 6

Two elements a and b in a principal ideal ring, R, are associates if and only if a = bu for some unit u.

Proof. If a = bu with u a unit then $au^{-1} = bu\,u^{-1} = b$ so that a|b and b|a. Therefore a and b are associates.

Now assume that a and b are associates. Then a = bc and b = ad so that a = adc. Since R is a principal ideal ring the cancellation law holds and 1 = dc so that d and c are units.

DEFINITION

If a and b are two elements of R then an element $d \in R$ is a greatest common divisor of a and b provided:

(1) d|a and d|b.

(2) Whenever c|a and c|b, then c|d.

If d and d' are greatest common divisors of a and b then d and d' are associates.

Lemma 7

Let R be a principal ideal ring. Then any two elements $a, b \in R$ have a greatest common divisor d. Moreover any such greatest common divisor can be realized as $d = \alpha a + \beta b$ for some $\alpha, \beta \in R$.

Proof. Let $A = \{ra + sb : r, s \in R\}$. Obviously A is an ideal. Since R is a principal ideal ring there exists an element d of R such that (d) = A. Since $d \in A$, d is of the form $\alpha a + \beta b$ for some $\alpha, \beta \in R$. Because a = 1a + 0b and b = 0a + 1b, both $a, b \in A$ so that d|a and d|b, so that d is a common divisor.

Finally suppose that c is an element such that $c|a$ and $c|b$. Then $c|\alpha a$ and $c|\beta b$ so $c|(\alpha a + \beta b) = d$. Therefore d is a greatest common divisor.

DEFINITION

In a principal ideal ring R, a and b are relatively prime if any (hence all) of their greatest common divisors are units.

Lemma 8

Let R be a principal ideal ring. Suppose that for a, b, $c \in R$, $a|bc$ but that a and b are relatively prime. Then $a|c$.

Proof. Let the unit u be a greatest common divisor of a and b. From Lemma 7, $u = \alpha a + \beta b$ for some $\alpha, \beta \in R$. Since u is a unit u has an inverse, u^{-1}. Now $1 = u^{-1}u = \cdots = u^{-1}\alpha a + u^{-1}\beta b$. Multiplication by c yields $c = cu^{-1}\alpha a + cu^{-1}\beta b$. Now $a|cu^{-1}\alpha a$ and $a|cb\beta u^{-1}$ so that $a|(cu^{-1}\alpha a + cb\beta u^{-1}) = c$.

Lemma 9

If p is a prime element of a principal ideal ring and if $p|ab$, then $p|a$ or $p|b$.

Proof. Suppose $p \nmid a$; if x is a greatest common divisor of p and a, then $x|p$ and $x|a$. Since $x|p$, x is either a unit or an associate of p. Since $p \nmid a$, but $x|a$, x is a unit and p and a are relatively prime. Therefore from Lemma 8, we have that $p|b$.

Lemma 10

Let R be a principal ideal ring, and suppose that a, b ϵ R. Then a|b if and only if (b) $\subseteq$ (a) with equality if and only if a and b are associates.

Proof. Suppose that a|b. Then b = na for some n ϵ R. Let r ϵ(b). Then r = mb = mna for some m ϵ R so that r ϵ(a) and (b) $\subseteq$ (a).

Conversely, suppose that (b) $\subseteq$ (a). Then b ϵ(b) $\subseteq$(a) so that b ϵ(a) and b = na for some n ϵ R. Thus a|b.

Now a and b are associates if and only if a|b and b|a, so that a and b are associates if and only if (a) $\subseteq$ (b) and (b) $\subseteq$ (a), i.e. if and only if (a) = (b).

Theorem 1

Let R be a principal ideal ring. An ideal P of R is prime if and only if any (hence every) generator of P is a prime element.

Proof. Let p be a generator of P, i.e. P = (p), and assume p = ab. We shall show that a or b must be a unit in R. Since p = ab ϵ P and P is prime, we must have a or b ϵ P. Assume that a ϵ P. Then a = np for some n ϵ P. Substituting for a we obtain p = bnp. Using the cancellation law (R is an integral domain), we have that 1 = bn, and b is a unit. Therefore P is prime.

Conversely suppose that p is a prime element of R and consider the ideal (p). If ab ϵ(p) we have p|ab. From Lemma 8 it follows that p|a or p|b since p is prime and p|ab.

If we assume that $p|b$. Then $b = np$ for some $n \in R$ and $b \in (p)$. Therefore (p) is prime.

Theorem 2.

Let R be a principal ideal ring. Then an ideal M of is maximal if and only if any (hence every) generator of M is a prime element of R.

Proof. First let us assume that p is a generator of and that p is not a prime element. Then $p = ab$ with a, b nonunits of R. Lemma 10 tells us that since $a|p$ and a is not an associate of p, then $(p) \subset (a)$ and $(p) = M$ is not maximal.

Conversely let us assume that p is a prime element, and that $(p) \subseteq I$ for some ideal I of R. Since R is a principal ideal ring, $I = (x)$ for some $x \in R$. But again using Lemma 10, $(p) \subseteq (x)$ tells us that $x|p$ so that $p = nx$ for some $n \in R$. Since p is prime either n or x is a unit. If n is a unit, then x and p are associates and $(x) = (p)$. I x is a unit, $(x) = R$, so that (p) is maximal.

Corollary: If R is a principal ideal ring, then an ideal is maximal if and only if it is prime.

We now show that if R is an arbitrary ring (commutative with identity, but not necessarily a principal ideal ring), then every maximal ideal is prime, but every prime ideal need not be maximal.

Lemma 11

A necessary and sufficient condition that a ring R (commutative with identity) should be a field is that (0) and

should be its only ideals; i.e. a commutative ring with identity is a field if and only if it has only trivial ideals.

Proof. Suppose that R is a field and that $I \neq (0)$ is an ideal of R. Choose $a \in I$, $a \neq 0$. The inverse of a is in R since R is a field; since I is an ideal $aa^{-1} = 1 \in I$. But if $1 \in I$, it follows that $I = R$.

Now suppose that R has (0) and itself as its only ideals. Let $x \in R$ be a nonzero element and consider the ideal (x). This ideal $\neq (0)$ because $x \in (x)$, so that $(x) = R$. But this tells us that $xy = 1$ for some $y \in R$. Therefore every nonzero element of R has an inverse and R is a field.

Theorem 3

If R is a ring and if I is an ideal of R, then the quotient structure R/I which consists of all cosets a+I ($a \in R$) is a ring under the definitions:

$$(a+I) + (b+I) = (a+b) + I$$

$$(a+I)(b+I) = (ab) + I$$

Proof. A proof of the theorem is given in McCoy [5].

Lemma 12

An ideal I of a ring R (commutative with identity) is prime if and only if R/I is an integral domain.

Proof. Suppose that I is a prime ideal of R, and suppose that $(a+I), (b+I) \in R/I$, with $(a+I)\cdot(b+I) = ab+I = I$,

the zero element of R/I. Now ab+I = I implies that ab $\in$ I. Since I is prime, either a $\in$ I or b $\in$ I; i.e. either a+I = I or b+I = I, and hence R/I is an integral domain.

Conversely assume that I is an ideal of R such that R/I is an integral domain. Let ab $\in$ I and assume that a $\notin$ I. Then a+I $\neq$ I. But (a+I)$\cdot$(b+I) = ab+I = I, and since R/I is an integral domain, we have that (b+I) = I, i.e. that b $\in$ I, so that I is prime.

Theorem 4

Let M be an ideal of R. Then M is maximal if and only if R/M is a field.

Proof. Suppose that M is maximal, and suppose that a+M $\in$ R/M, where a+M $\neq$ M, i.e. a $\notin$ M. Since M is maximal, the ideal M+(a) generated by M and a must be R itself. Since 1 $\in$ M+(a), there exist m $\in$ M and r $\in$ R such that 1 = m +ra. But this implies that (r+M)$\cdot$(a+M) = ra+M = (1-m) + M = 1+M, the unit element of R/M; i.e. (a+M) is invertible. Since R is commutative, R/M is also commutative and is therefore a field.

Conversely, suppose that M is an ideal of R such that R/M is a field and suppose that I is an ideal of R such that M $\subset$ I. Choose a $\in$ I-M and consider (a+M) $\in$ R/M. Since a $\notin$ M, we know that a+M $\neq$ M so that (a+M) is invertible; i.e. there exists b+M $\in$ R/M such that (a+M)$\cdot$(b+M) = ab+M = 1+M, so that ab-1 $\in$ M $\subset$ I. But ab $\in$ I since a $\in$ I, so that 1 $\in$ I and I = R.

Corollary 1

In a ring R (commutative with identity) every maximal ideal is a prime ideal.

Proof. From Theorem 4, M is maximal if and only if R/M is a field. Lemma 12 tells us that if R/M is an integral domain, then M is prime. Since every field is an integral domain, every maximal ideal is prime.

The converse to the preceding corollary is false, for consider the following example.

Let R be the direct sum of two copies of Z; i.e. R consists of all ordered pairs of integers (a,b) under the component-wise operations. R is not an integral domain since $(1,0)\cdot(0,1) = (0,0)$. In fact R contains infinitely many zero divisors.

Now let I be the ideal of R consisting of all the elements of the form (n,0). Now I is a prime ideal for suppose $ab = (n,0) \in I$ and assume $a \notin I$. Since $a \in I$, $a = (m,i)$ where $i \neq 0$. If $b = (p,j)$, then $ji = 0$ so that $j = 0$, i.e. $b \in I$ and I is prime.

But I is not maximal for consider the ideal $J = \{(m,2n) : m,n \in Z\}$. Then $I \subset J \subset R$ and thus I is not maximal.

The prime elements of R are all of the form $(\pm 1, \pm p)$ or $(\pm p, \pm 1)$ with p a prime, and the units are $(\pm 1,0)$ and $(0,\pm 1)$. Every ideal of R generated by a prime element is maximal, and hence prime, and conversely every maximal ideal is generated by a prime element. In addition every ideal of R is principal.

We earlier noted that the ring C(I) has no prime elements. This ring is of interest in the study of maximal and prime ideals for two other reasons. First it is another example of a ring in which prime ideals are not necessarily maximal, and second its maximal ideals are not necessarily principal.

III. MAXIMAL IDEALS IN C(X), THE RING OF CONTINUOUS FUNCTIONS ON A TOPOLOGICAL SPACE X.

Lemma 13

Let I be an ideal of a ring R (commutative with identity). If S is any subset such that $S \cap I = \phi$, then there exists an ideal P of R such that:

(1) $I \subseteq P$

(2) $P \cap S = \phi$

(3) The ideal P is maximal with respect to properties (1) and (2). I.e. if Q is any ideal such that $Q \supseteq P$ and $Q \cap S = \phi$, then $Q = P$.

Moreover, if S is closed under multiplication, then P is a prime ideal.

Proof. By the maximal principle, there exists a maximal chain C of ideals containing I and disjoint from S. Let $P = \cup C$; then from Lemma 3 P is an ideal containing I, disjoint from S, and maximal with respect to this property.

Assume now that S is closed under multiplication, and let a and b be any two elements of R - P; consider the ideals P+(a) and P+(b). Because of the maximality of P

these ideals must contain elements of S. Let $s \in (P+(a)) \cap S$ and $t \in (P+(b)) \cap S$. Then $s = xa+p$ and $t = yb+q$ with $x,y \in R$ and $p,q \in P$. Now consider $st = xyab + ybp + xaq + pq$. Since S is closed under multiplication, $st \in S$. But $S \cap P = \phi$ so that $st \notin P$. But ybp, xaq, and pq are in P so that their sum is in P. Therefore $xyab \notin P$ (if so we would have $st \in P$). Therefore $ab \notin P$ and P is prime.

Corollary 1

Let I be an ideal. If no power of a belongs to I, then there exists a prime ideal containing I but not a.

Proof. Let $S = \{x:x = a^n, n = 1,2, \ldots\}$. Then S is closed under multiplication, and $a \notin I$, $I \cap S = \phi$. Now apply the preceding lemma.

Corollary 2

The intersection of all the prime ideals containing a given ideal I is precisely the set of all elements x such that $x^n \in I$ for some $n \geq 1$.

Proof. If there exists a prime ideal P containing I but not a, then no power of a can belong to I, since no power of a belongs to P. Conversely if no power of a belongs to I, then by preceding corollary, some prime ideal contains I but not a.

Corollary 3

If R is a commutative ring with identity which is not a field, then R contains at least one maximal ideal.

Proof. Since R is not a field, R contains a non-trivial ideal I. Since I is proper, $1 \notin I$. Let $S = \{1\}$. If P is maximal with respect to the properties $P \cap S = \phi$ and $I \subseteq P$, then P is a maximal ideal in the usual sense.

Corollary 4

In a commutative ring with identity, every maximal ideal is prime.

Proof. Let I be a maximal ideal in R. Since I is proper $1 \notin I$ so that $I \cap S = \phi$, where $S = \{1\}$. But S is closed under multiplication and I is maximal with respect to $I \cap S = \phi$, so that I is prime.

Example

Let $M_o = \{f \in C(R) : f(0) = 0\}$, i.e. M_o consists of all functions in C(R) which vanish at the origin. We claim M_o is a maximal ideal, for consider an ideal I such that $M_o \subset I$. Let $g \in I - M_o$. Then $g(0) \neq 0$. Since $g \in I$, $gg = g^2 \in I$ and g^2 is everywhere non-negative. The function $i(x) = x$ is in M_o and therefore in I. Let $h = g^2 + i$. The function h is everywhere > 0 and is therefore invertible, so that $I = C(R)$. Thus M_o is a maximal ideal.

We shall now use Corollary 1 of Lemma 13 to exhibit a prime ideal which is properly contained in M_o. First let us consider the principal ideal generated by the function $i(x) = x$. A function $f \in (i)$ if and only if $f = ig$ for some $g \in C(R)$. The ideal (i) is not prime, for $i \in (i)$ and $i = i^{1/2} i^{1/2}$ where $i^{1/2}(x) = \sqrt{x}$. Now $i^{1/2} \notin (i)$ for if so we would have

$i^{1/2} = gi$ for some $g \in C(R)$. But then by continuity $g(0) = \lim_{x \to 0+} g(x) = \lim_{x \to 0+} \sqrt{x}/x = \lim_{x \to 0+} 1/\sqrt{x} = +\infty$, a contradiction. Therefore (i) is not prime.

If we can find a function f such that $f \in M_o$, but no power of f is in (i), by Corollary 1 of Lemma 13 we know that there exists a prime ideal P containing (i) but not f so that $P \subset M_o$. Let us consider the following function:

$$f(x) = \begin{cases} 0 & x = 0 \\ -1/\log x & 0 < x < 1/2 \\ -1/\log 1/2 & 1/2 \leq x \end{cases}$$

We want to show that for no $n \geq 1$ is $f^n \in (i)$. Suppose, to the contrary, that for some n it is true that $f^n \in (i)$, i.e. that $f^n = ig$, for some $g \in C(R)$. Then by continuity, $g(0) = \lim_{x \to 0+} g(x) = \lim_{x \to 0+} 1/x|\log x|^n$. By repeated application of l'Hopital's rule, it is seen that

$$\lim_{x \to 0+} 1/x|\log x|^n = +\infty$$

and the contradiction is reached.

The interval [0,1] is a particular example of a compact topological space. The next theorem concerning maximal ideals of functions on general compact spaces will be stated in two ways. The first statement is proven directly and requires no new concepts. For the second statement we will need the ideas of fixed and free ideals. The second statement will be instructive in the discussion of maximal ideals

of rings of continuous functions on an arbitrary (not necessarily compact) topological space. With this in mind we prove:

Theorem 5

Let C be the ring of continuous real-valued functions on a compact topological space X. The the maximal ideals of C are precisely the sets

$$M_p = \{f \in C : f(p) = 0\}.$$

Proof. Fix $p \in X$ and consider the homomorphism φ which maps C onto the real line R, defined by $\varphi(f) = f(p)$. It is obvious that φ is onto R for consider $r \in R$; the constant function $f_r(x) \equiv r$ for all $x \in X$ is in C for all $r \in R$ and $\varphi(f_r) = f_r(p) = r$. The kernel of φ is just the set M_p defined above. Since φ maps C onto R, it follows from the fundamental theorem of ring homomorphisms [4] that $C/M_p \approx R$: since R is a field, the ideal M_p must be maximal (Theorem 4).

The elements of C/M_p are the cosets $f+M_p$. We point out that for every coset $f+M_p \in C/M_p$ there exists a constant function $g \in C$ such that $f+M_p = g+M_p$. Define g by $g(x) \equiv f(p)$ for all $x \in X$. Then $\varphi(f-g) = \varphi(f)-\varphi(g) = f(p)-g(p) = f(p)-f(p) = 0$ so that $f-g \in M_p$, i.e. $f+M_p = g+M_p$.

Now suppose that M is a maximal ideal which is not of the form M_p for any p. Then for any $x \in X$, there exists $f_x \in M$ such that $f_x(x) \neq 0$, for if there existed an $x_o \in X$ such that $f(x_o) = 0$ for every $f \in M$, then either $M \subset M_p$, in

which case M is not maximal or $M = M_p$ and that contradicts our assumption that for no p is M of the form M_p. Now consider any $x \in X$ and consider $f_x \in M$ such that $f_x(x) \neq 0$. Since f_x is continuous, there exists a neighborhood N(x) of x such that $f_x(t) \neq 0$ for all $t \in N(x)$. The family $\mathcal{N} = \{N(x) : x \in X\}$ covers X; since X is compact, we can extract a finite subcover $\{N(x_1), \ldots, N(x_k)\}$ which also covers X. Let $f_1 = f_{x_1}$, etc. Then $g = f_1^2 + f_2^2 + \ldots + f_k^2 \in M$ and moreover for every $x \in X$, $g(x) > 0$ so that g is invertible and hence M = C.

DEFINITION

Let X be an arbitrary topological space. Then C(X) will denote the ring of continuous real-valued functions on X under the usual pointwise operations and C*(X) will denote the subring of bounded continuous functions. If X is compact, then C(X) = C*(X); conversely a space in which every continuous function is bounded and therefore for which C and C* are the same is called pseudocompact. Pseudocompactness does not imply compactness.

Example

Consider the space X = (0,1) with a topology of all the sets of the form (0,a) where $0 \leq a \leq 1$. ($(0,0) = \phi$.) Obviously X is not compact. Now let $f \in C(X)$ and assume that x, y are distinct points in the range of f, i.e. $x \neq y$. Now let F_1 and F_2 be the pre-images of x and y respectively. Since f is continuous, F_1 and F_2 are closed sets with

Example

(1) As an example, let R be the real line under its usual topology and consider C(R). If I is the ideal

$$I = \{f \in C(R) : f(0) = 0\},$$

then evidently $\cap Z[I] = \{0\}$, so that I is fixed.

(2) A function is of compact support if it vanishes outside a closed, bounded interval. If K is the ideal of functions of compact support, then $\cap Z[K] = \phi$ so that K is free.

We shall now restate Theorem 5 in a slightly different manner. Since the proof of the first part of the theorem is the same as before we omit it.

DEFINITION

A topological space is a Hausdorff space if whenever x and y are distinct points of the space then there exist disjoint neighborhoods of x and y. I.e. there exist neighborhoods N_x of x and N_y of y such that $N_x \cap N_y = \phi$.

DEFINITION

A Hausdorff space X is said to be completely regular if whenever F is a closed set and x is a point of $X \sim F$, there exists a function $f \in C(X)$ such that $f(x) = 1$ and $f(F) = \{0\}$. In a completely regular space if $f(x) = f(y)$ for all $f \in C(X)$, then $x = y$.

Theorem 5

Let X be a completely regular space. The fixed maximal ideals in C(X) are precisely the sets

$$M_p = \{f \in C(X) : f(p) = 0\}.$$

The ideals M_p are distinct for distinct p. For each p, $C(X)/M_p$ is isomorphic to R, the field of real numbers.

Proof. We shall prove only that a fixed ideal is maximal only if it is equal to a set M_p, as stated above. Let M be any fixed ideal. Since M is fixed there exists at least one point $p \in \cap Z[M]$. Since every $f \in M$ vanishes at p, $M \subseteq M_p$ and M is maximal if and only if $M = M_p$.

Except for notation the theorem is identical for the subring C*(X).

Corollary

If X is compact, then every ideal I in C(X) is fixed.

Proof. The corollary follows immediately from the second part of the first proof of Theorem 5. I.e. if X is compact and $\cap Z[I] = \phi$, then I contains a function of the form $g = f_1^2 + f_2^2 + \ldots + f_k^2$ with $g > 0$ everywhere so that g is a unit and I = C.

In a principal ideal ring we proved that the maximal ideals were generated by prime elements. We have seen that C(X) contains no prime elements. We have also shown that in the general case maximal ideals are not necessarily principal. There is a special case for which the maximal ideals are principal. This occurs when X is a discrete space.

DEFINITION

A point p of a topological space X, is isolated if the set $\{p\}$ is open. A space is discrete if every point is isolated.

Lemma 14

Suppose that X is a completely regular space, and suppose that $p \in X$ is such that the ideal $M_p = \{f \in C(X): f(p) = 0\}$ is principal. Then p is isolated; i.e. $\{p\}$ is open. Conversely, if p is isolated M_p is principal.

Proof. Suppose that p is isolated and let M_p be as above. Consider the following f defined by $f(x) = 1$ if $x \neq p$ and $f(p) = 0$. Then obviously $f \in C(X)$ since p is isolated. Also $f \in M_p$. But if $g \in M_p$, then $g = gf$ so that $M_p = (f)$ is principal.

Conversely suppose that M_p is principal, but that $\{p\}$ is not open. Since $\{p\}$ is not open, $p \in \overline{X-\{p\}}$, so that there exists a net $\{x_\alpha\}$ of elements of $X-\{p\}$ such that $x_\alpha \to p$. Since M_p is principal, $M_p = (f)$ for some $f \in C(X)$. We claim $f(x) = 0$ if and only if $x = p$. Suppose that $f(q) = 0$ for some $q \in X$, $q \neq p$. Then $g \in M_p$ implies $g(q) = f(q) = 0$. But since X is Hausdorff, there exists a neighborhood N(p) such that $q \notin N(p)$. Since X is completely regular, there exists $h \in C(X)$ such that $h(p) = 0$ and $h(x) = 1$ if $x \notin N(p)$, i.e. $h(q) \neq 0$. But $h(p) = 0$ implies that $h \in M_p$, so that $h(q) \neq 0$ is a contradiction. Consequently f vanishes only at p.

Consider now $f^{\frac{1}{3}} \in C(X)$. Since $f^{\frac{1}{3}}(p) = \sqrt[3]{f(p)} = 0$, $f^{\frac{1}{3}} \in M_p$, so that $f^{\frac{1}{3}} = fg$ for some $g \in C(X)$. But then $g(x_\alpha) = f^{\frac{1}{3}}(x_\alpha)/f(x_\alpha) = (f(x_\alpha))^{-\frac{2}{3}}$, which is defined since $f(x_\alpha) \neq 0$. Since g is continuous, $g(p) = \lim g(x_\alpha) = \lim[f(x_\alpha)]^{-\frac{2}{3}} = +\infty$, a contradiction.

Corollary

Every fixed maximal ideal in C(X) is principle if X is a discrete space.

Proof. Since every fixed maximal ideal is of the form M_p, for some $p \in X$, and since every $x \in X$ is isolated if X is discrete, it follows directly from Lemma 14 that $M_p = (f)$ for some $f \in C(X)$.

Since the real line R, is an example of a non-compact space, let us have a closer look at the free ideals of C(R) and C*(R).

We earlier noted that the ideal K, consisting of those functions of C(R) which vanished outside a compact subset of R was a free ideal. We now show that K is not maximal. In fact K is contained in an uncountable number of maximal ideals each of which is free.

Let S be any infinite subset of R which has no limit points. (Thus S is unbounded and countable.) Let J(S) be the following set: $J(S) = \{f \in C(R) : f(s) \neq 0$ for at most finitely many $s \in S\}$. It can be easily verified that J(S) is a free ideal, and that $J(S_1) \supseteq J(S_2)$ if and only if $S_1 - S_2$ is finite. Thus $J(S_1) \supset J(S_2)$ if and only if $S_1 - S_2$ is finite and $S_2 - S_1$ is infinite.

Let $N \subset R$ be the set of positive integers and consider $J(N)$. Any function in K must be in $J(N)$, but let $f(x)=\sin(\pi x)$. Since $f(n) = 0$ for all $n \in N$ it follows that $f \in J(N)$. Obviously $f \notin K$ so that $K \subset J(N)$. Now let p be a prime number and let

$$P_o = \{p,2p,3p, \ldots ,np, \ldots\}$$
$$P_1 = \{2p,4p,6p,\ldots ,2np,\ldots\}$$
$$P_2 = \{4p,8p,12p, \ldots , 4np, \ldots\}$$
$$\ldots$$
$$P_k = \{2^k p,2\cdot 2^k p, \ldots , n2^k p,\ldots\}$$
$$\ldots .$$

Then $K \subset J(N) \subset J(P_o) \subset J(P_2) \subset \ldots \subset J(P_k) \subset \ldots$, and we have a chain of free ideals. Now consider the set $T = \{p^{2^n} : n = 1,2,\ldots\}$. We introduce T to show that the ideal $\cup J(P_n)$ $n = 1,2,3,\ldots$ is not maximal. We claim that $\cup J(P_n) \subset J(T)$ for let $f \in \cup J(P_n)$. Then $f \in J(P_k)$ for some k, so that $f(x) \neq 0$ for only finitely many $x \in P_k$. Since $T-P_k$ is finite, $f(y) \neq 0$ for only finitely many $y \in T$, and therefore $f \in J(T)$. Now let $g \in J(T)$ be such that $g(y) = 0$ for all $y \in T$, but $g(y) \neq 0$ for all y of the form $2^n q$ for all odd $q \geq 3$ and all $n \geq 1$. Then $g \notin \cup J(P_n)$, so that $\cup J(P_n) \subset J(T)$. If we now consider

$$T_2 = \{p^{2^{2^n}} : n = 1,2,3, \ldots\}$$

$$T_3 = \{p^{2^{2^{2^n}}} : n = 1,2,3, \ldots\}$$

. . .

$$T_k = \{p^{2^{2^{\cdot^{\cdot^{\cdot^{n}}}}}} : n = 1,2,3, \ldots\}$$

we see that $J(T) \subset J(T_2) \subset \ldots \subset J(T_k) \subset \ldots$. The maximal principle assures us of the existence of a maximal ideal containing the above ideals, but one sees that this maximal ideal cannot be readily displayed.

We note that if q is a prime such that $p \neq q$, then $J(Q)$ is contained in a maximal ideal different from that containing $J(P)$. Assume not. Let M be any ideal containing $J(P)$ and $J(Q)$. Then $f(x) = \sin(\pi x/q)$ is in $J(Q)$ and $g(x) = \sin(\pi x/p)$ is in $J(P)$. Hence $h = f^2+g^2 > 0$ for all $x \in R$ and $h \in M$ is a unit, and $M = C(R)$.

By considering a Hamel basis for R it can be shown that $C(R)$ contains uncountably many free, maximal ideals.

Let us now consider the free maximal ideals of $C^*(R)$. Let $I = \{f \in C^*(R) : \lim_{x\to\infty} f(x) = 0\}$. Since $C^*(R)$ consists of bounded functions, I can easily be shown to be an ideal. Obviously I is free since if we let $h(x) = 1/(1+x^2) > 0$ for all $x \in R$ then $\lim_{x\to\infty} h(x) = 0$, so that $h \in I$. Further I is proper since if $f \in I$, then either $f(x_o) = 0$ for some x_o and f is not invertible, or if $f(x) \neq 0$ for all x, then $1/f$ is not bounded and thus f is not invertible in $C^*(R)$.

We claim that I is not maximal for let S be an infinite subset of R which contains no limit points and consider $J^*(S) = \{f \in C^*(R) : f(s) \neq 0$ for only finitely many $s \in S\}$. Obviously $J^*(S)$ is an ideal. Now consider $M = I+J^*(S)$. From Lemma 4 we know M is the ideal generated by I and $J^*(S)$. To see that $M \neq C^*(R)$, consider $f+g \in M$, and let $\epsilon > 0$. Then there exists an $N > 0$ such that for all $|x| > N$, it is true that $|f(x)| < \epsilon$. Now let $S_1 = S \cap [(-\infty,-N) \cup (N,\infty)]$. Since $g \neq 0$ for only finitely many $s_1 \in S_1$, there exists points $y \in S_1$ such that $g(y) = 0$, i.e. such that $f(y)+g(y) < \epsilon$. Thus $h = f+g$ is not bounded away from 0 and so h is a non-unit in $C^*(R)$; therefore $M \neq C^*(R)$.

The procedure for showing the existence of uncountably many free maximal ideals in C(R) relied on showing that $\sin^2(\pi x/p) + \sin^2(\pi x/q) > 0$ for all $x \in R$ and was therefore invertible. It is not as easy in $C^*(R)$ since we must show that a function is bounded away from 0 if it is invertible.

DEFINITION

Let X be a non-compact topological space, and let X^* be a compact space such that $X \subset X^*$ and such that the inclusion $i: X \to X^*$ is a homeomorphism. Then X^* is a compactification of X if X is dense in X^*.

There are several methods by which non-compact spaces can be compactified. For example the interval (0,1) under its usual topology can be compactified by the addition of

the points 0 and 1 to obtain the compact space [0,1]. Another method is the Stone-Čech compactification process [2]. Still another is the one-point compactification of Alexandroff.

DEFINITION

A topological space is locally compact if for every $x \in X$ there is an open set G containing x such that the closure of G is compact.

Examples

(1) Every discrete space is locally compact, therefore N, the space of positive integers is locally compact.

(2) The Euclidean spaces R^n are locally compact. Any open interval (a,b) on the real line is locally compact.

Theorem 5 (Alexandroff One Point Compactification)

Let X be a locally compact Hausdorff space. Let $X^* = X \cup \{\infty\}$, where "∞" is any point not in X. Define a subset U of X^* to be open if it is either an open set in X or the complement of a compact subset of X. Then X^* is a compact, Hausdorff space and the inclusion mapping of X into X^* is a homeomorphism, so that X^* is a compactification of X.

Proof. A proof of this theorem is given in Kelley[3]

If X is a locally compact Hausdorff space and if $f \in C(X)$, we say that f has a limit at infinity if for all $\epsilon > 0$, there exists a compact subset K_ϵ such that $x,y \in X-K_\epsilon$ implies that $|f(x)-f(y)| < \epsilon$. It is easy to see that if

$g \in C(X^*)$, then $g|X \in C(X)$ which implies that $g|X$ has a limit at infinity (namely $g(\infty)$); conversely if $h \in C(X)$ has a limit at infinity h can be extended to $h^* \in C(X^*)$ by defining $h(\infty)$ to equal this limit. Thus there exists a one-one correspondence between $C(X^*)$ and those functions in $C(X)$ which have limits at infinity, so that $C(X^*)$ is embedded in $C(X)$.

By Corollary 1 to Theorem 5, we know that $C(R^*)$ and $C^*(R^*)$ contain only fixed ideals. One might naturally ask what happened to the free ideals in the compactification. We note that the ideal K is no longer free in the compactification since $\cap K = \{\infty\}$. We further note that the functions in $C(R)$ which have no limit at infinity are not carried over into $C(R^*)$, and therefore periodic functions of constant amplitude such as $\sin(\pi x/p)$ are not in $C(R^*)$. If we consider an infinite set S with no limit points, it is true that if $f \in J(S)$ and if f has a limit at ∞, then this limit must be zero; therefore all of the ideals $J(S)$ collapse into M_∞.

In reality the compactification of a space such as R does not enlighten us toward characterization of the free maximal ideals. We know only that an arbitrary non-compact space can be compactified and that the maximal ideals of a compact space are fixed. Moreover these ideals are in a natural one-to-one correspondence with the points of the space.

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DOCUMENT CONTROL DATA - R & D

Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1 ORIGINATING ACTIVITY (Corporate author)
Naval Postgraduate School
Monterey, California 93940

2a. REPORT SECURITY CLASSIFICATION
UNCLASSIFIED

2b GROUP

3 REPORT TITLE

CHARACTERIZATION OF MAXIMAL IDEALS

4 DESCRIPTIVE NOTES *(Type of report and, inclusive dates)*
M.S. Thesis - Mathematics - 1968

5 AUTHOR(S) *(First name, middle initial, last name)*
John D. Carpenter
Lieutenant, United States Navy

6 REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
December 1968	41	6

8a. CONTRACT OR GRANT NO	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d.	

10. DISTRIBUTION STATEMENT
[redacted]s
[redacted]
[redacted]
[redacted]graduate School [redacted]

11 SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
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13 ABSTRACT

The characterization of maximal ideals has been accomplished for the class of principal ideal rings and for the class of continuous functions on a compact topological space. In the latter case the characterization is somewhat more extensive than it appears since a non-compact space can often be compactified. Closely related to maximal ideals are the prime ideals and the coincidence of these two types of ideals is noted where occurring. When the prime and maximal ideals are not the same in a ring, illustrative examples are given.

DD FORM 1 NOV 65 1473 (PAGE 1)
S/N 0101-807-6811

14 KEY WORDS	LINK A		LINK B		LINK C	
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Prime Ideals						
Principal Ideal Rings						
Rings of Continuous Functions						

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